Constructing Ideal Secret Sharing Schemes based on Chinese Remainder Theorem

Yu Ning, Fuyou Miao,…*

University of Science and Technology of China

Contributions

- ✓ Generalization of existing CRT-based (t,n)-SS from Integer Ring to Polynomial Ring
- ✓ Ideal (t,n)-SS based on CRT for Poly. Ring
- ✓ Shamir's (t,n)-SS : a special case
- ✓ Weighted (t,n)-SS

Outline

- (t,n)-Threshold Secret Sharing (i.e., (t,n)-SS)
- Two Typical Secret Sharing Schemes
- Secret Sharing based on Polynomial Ring over F_p
- Both Types of SS as Special Cases
- Weighted (t,n)-SS
- Conclusion

(t,n)-Threshold Secret Sharing

- t-Threshold, n- number of all shareholders
- A dealer divides a secret s into n pieces, allocates each piece to a shareholder as the share such that
  - 1) any t or more than t shares can recover the secret;
  - 2) less than t shares cannot obtain the secret;

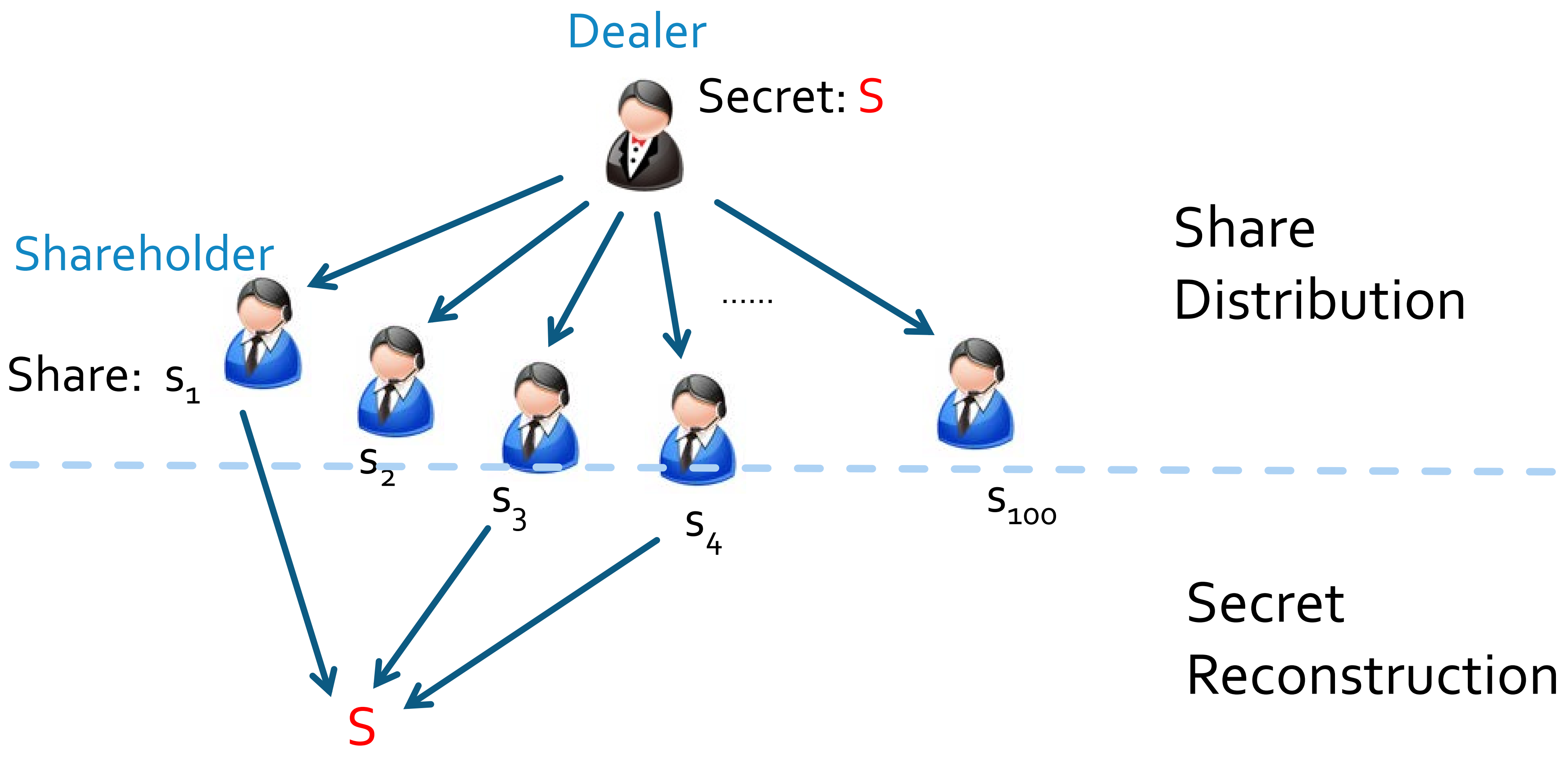


Fig 1. An example of (3,100)-SS

Applications of (t,n)-SS

- ✓ Threshold Encryption
- ✓ Threshold Signature
- ✓ Secure Multiparty Computation
- ✓ Many security-related application protocols…

2 Typical (t,n)-SSs

- Shamir's (t,n)-SS [23]*
  - *Share Distribution*
    - $f(x)=a_0+a_1x+a_2x^2+\ldots+a_{t-1}x^{t-1} \bmod p$

      Secret: $s=a_0$
    - *Each Shareholder U_i: Public information-$x_i \in F_p$, private share--$f(x_i)$*
  - *Secret Reconstruction*
    - *m （m≥t） shareholders, e.g. $\{U_1,U_2,\ldots U_m\}$, compute the secret as:*

$$s=\sum_{i=1}^{m} f(x_i)\prod_{\substack{j=1,\\ j\neq i}}^{m}\frac{x_j}{x_j-x_i} \bmod p, \qquad (m\geq t)$$

*[23] Shamir, A.: How to share a secret. Communications of the ACM **22**(11), 612-613 (1979)

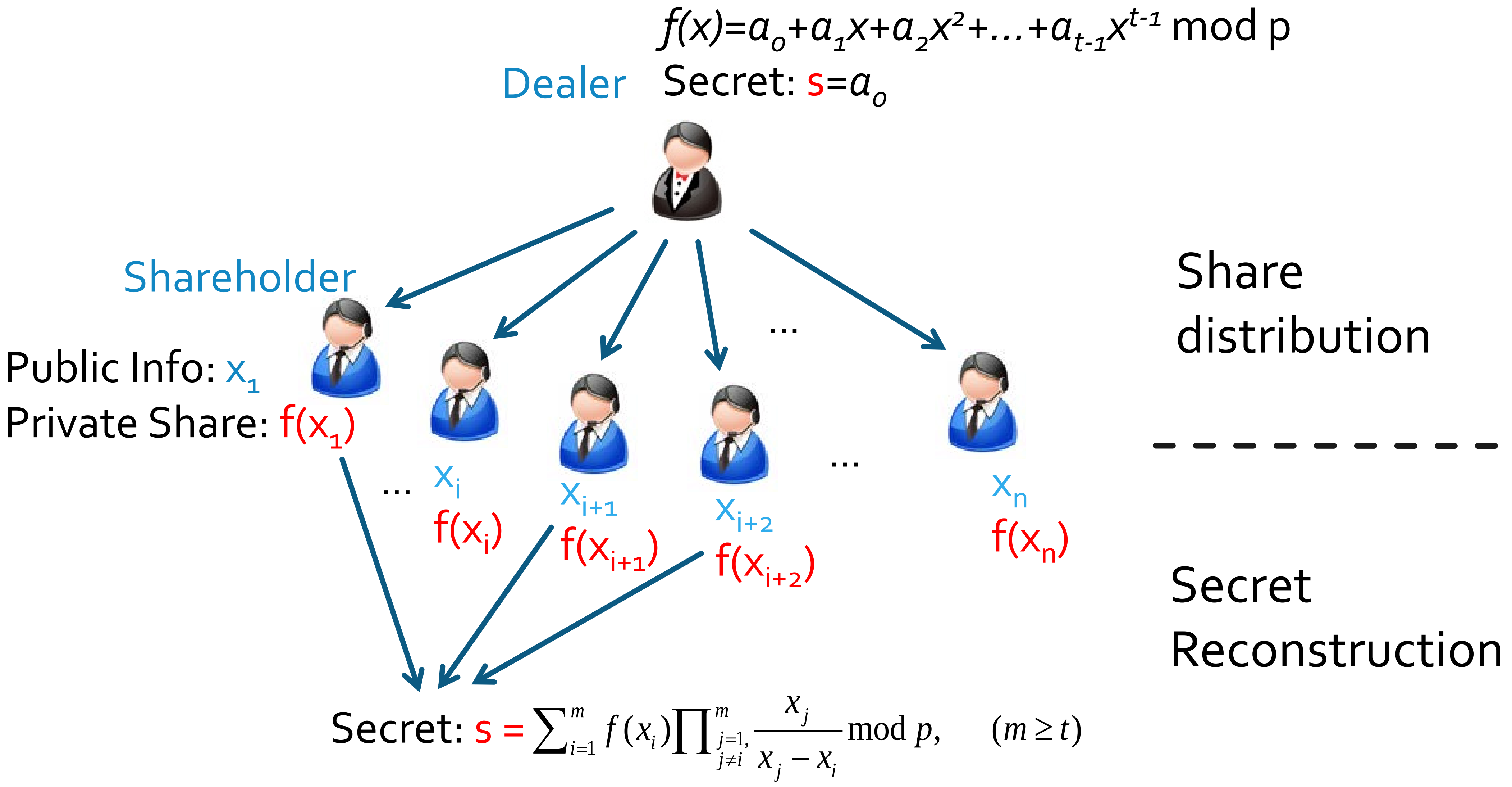


Fig 1. Shamir's (t,n)-SS

✓ Remarks

- Shamir's (t,n)-SS uses Lagrange Interpolation over finite field F_p to recover the secret.
- Ideal scheme:
  - Information rate 1;
  - No information leaks to t-1 participants
- Most popular (t,n)-SS scheme cited over 13000 times -- google scholar

2 Typical (t,n)-SS

✓ Asmuth-Bloom's (t,n)-SS[1] over 600 times of citation

- Share distribution:

secret : $s \in Z_{m_0}$, modulus of shareholder$_i$: $m_i \in Z$

$m_0 < m_1 < m_2 < \ldots < m_n, \gcd(m_i, m_j) = 1,$ (Increasing sequence, pairwise coprime)

$m_0 m_{n-t+2} \bullet \ldots \bullet m_n \le m_1 m_2 \bullet \ldots \bullet m_t \quad (*)$ (gap creation)

$B = s + \alpha m_0 < m_1 m_2 \bullet \ldots \bullet m_t$ (range extension)

$s_i = B \bmod m_i;$ (share evaluation)

[1]Asmuth, C., Bloom, J.:A modular approach to key safeguarding. IEEE transactions on information theory 29(2), 208-210 (1983)

- Secret Reconstruction

For authorized subset A, $|A| \geq t$ $\quad B = \sum_{i \in A} s_i \frac{M}{m_i} (\frac{M}{m_i})^{-1} \bmod m_i \bmod M$

secret: $s = B \bmod m_0$;

✓ Remark:

- Based on Chinese Remainder Theorem(CRT) for Integer Ring
- Not Ideal—information rate < 1
- Hard to choose moduli due to the condition

$$m_0 m_{n-t+2} \bullet \ldots \bullet m_n \leq m_1 m_2 \bullet \ldots \bullet m_t \qquad (*)$$

✓ Awkward scheme→[13-20][33]...

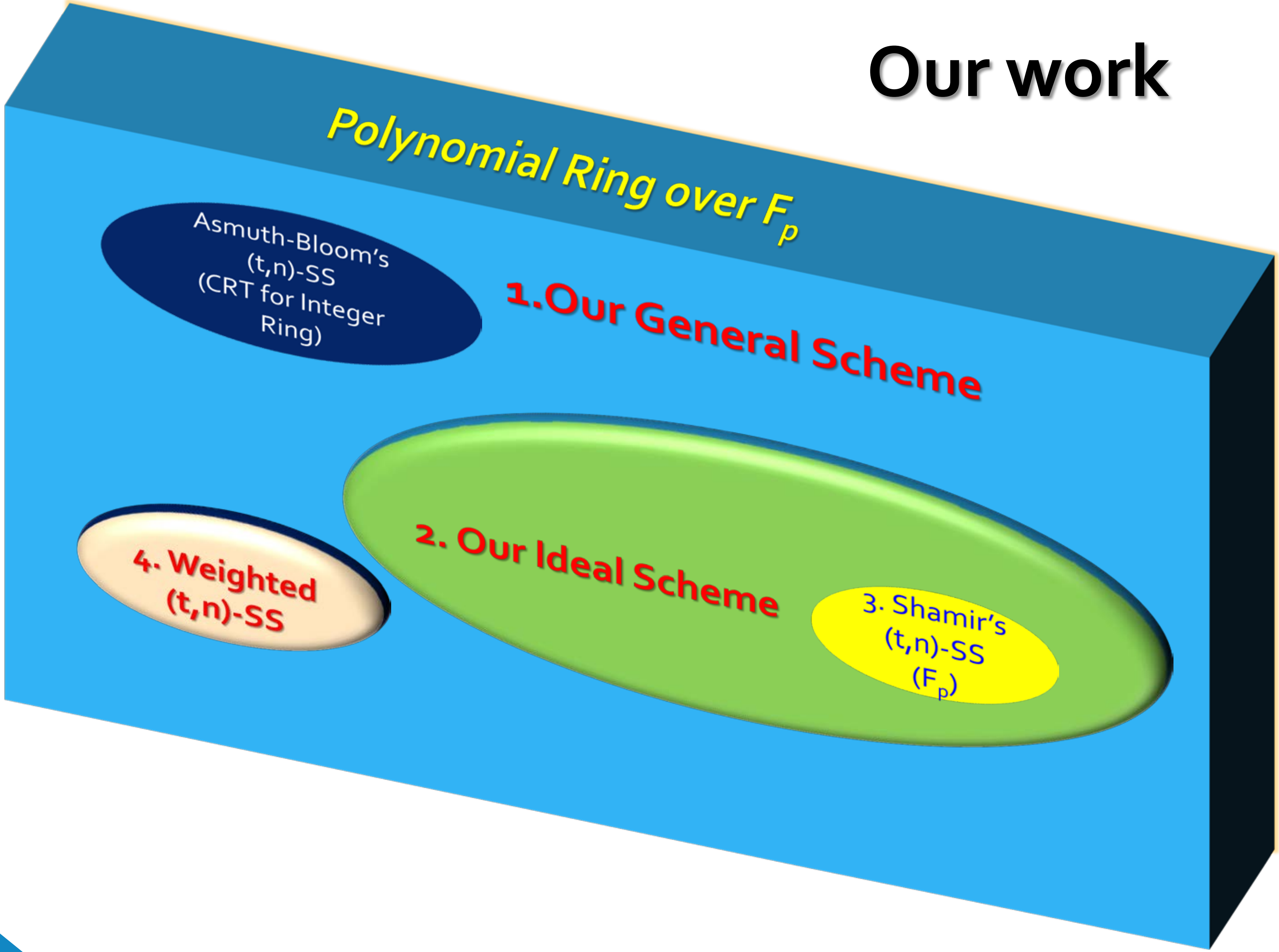
Questions

- ✓ Can we use CRT to build a (t,n)-SS as ideal as Shamir's scheme?
- ✓ What is the **connection** between **CRT based (t,n)-SSs** and **Shamir's (t,n)-SS**?

Our work

- ✓ Generalize Asmuth-Bloom's (t,n)-SS from Integer Ring to Polynomial Ring
  - ***General Scheme***
  - ***Ideal Scheme***
  - Prove Shamir's (t,n)-SS is a ***special case*** of our ***Ideal Scheme***
  - Construct a ***weighted (t,n)-SS*** from ***General Scheme***

Our work


Polynomial Ring over F_p
Asmuth-Bloom's (t,n)-SS (CRT for Integer Ring)
1.Our General Scheme
2. Our Ideal Scheme
3. Shamir's (t,n)-SS (F_p)
4. Weighted (t,n)-SS

(t,n)-SS based on CRT for Polynomial Ring over F_p

- ✓ General scheme
- ✓ Ideal scheme

Our General Scheme

✓ Setup

prime p, an integer $d_0 \geq 1$, $m_0(x) = x^{d_0}$,

pairwise coprime polynomials $m_i(x) \in F_p[x]$,

$d_i = \deg(m_i(x))$ for $i \in [0, n]$ such that

$d_0 \leq d_1 \leq d_2 \leq \ldots \leq d_n$ and $d_0 + \sum_{i=n-t+2}^{n} d_i \leq \sum_{i=1}^{t} d_i$ (*ascending sequence, gap production*)

✓ Share Distribution

The Dealer pick secret $s(x)$, $\deg(s(x)) < d_0$, random $\alpha(x)$, such that

$f(x) = s(x) + \alpha(x) m_0(x)$, $\deg(\alpha(x)) + d_0 < \sum_{i=1}^{t} d_i - 1$

share for ith shareholder:

$$s_i(x) = f(x) \bmod m_i(x)$$

Our General Scheme

✓ Secret Reconstruction

any k participants, e.g., $\{1, 2, \ldots, k\}$, $k \geq t$ recover the secret $s(x)$:

$$\begin{cases} f(x) = s_1(x) \bmod m_1(x) \\ f(x) = s_2(x) \bmod m_2(x) \\ \ldots \\ f(x) = s_k(x) \bmod m_k(x) \end{cases} \rightarrow \quad f(x), \quad \text{(by CRT for polynomial ring)}$$

$$\rightarrow \; s(x) = f(x) \bmod m_0(x)$$

Our Ideal Scheme

✓ Only Difference in Setup

prime p, an integer $d_0 \geq 1$, $m_0(x) = x^{d_0}$,

pairwise coprime polynomials $m_i(x) \in F_p[x]$,

$d_i = \deg(m_i(x))$ for $i \in [0, n]$ such that

$$d_0 = d_1 = d_2 = \ldots = d_n \text{ and } d_0 + \sum_{i=n\text{-}t+2}^{n} d_i = \sum_{i=1}^{t} d_i$$

$$d_0 \leq d_1 \leq d_2 \leq \ldots \leq d_n \text{ and } d_0 + \sum_{i=n\text{-}t+2}^{n} d_i \leq \sum_{i=1}^{t} d_i$$

(in general scheme)

Surprising Gains from Our Ideal Scheme

- ✓ Information rate=**1**, no info. leak →Ideal scheme
- ✓ Quite easy to choose pairwise coprime modulus polynomials
  - e.g. $x^{d_0}+1,\ \ x^{d_0}+2,\ldots,\ \ x^{d_0}+n$
- ✓ Shamir's (t,n)-SS as a special case

Shamir's (t,n)-SS as our special case

- ✓ An instantiation of our ideal scheme with

$$d_o=1$$

Shamir's (t,n)-SS as our special case

CRT for Polynomial Ring over F_p

$$\begin{cases} f(x) = s_1(x) \bmod m_1(x) \\ f(x) = s_2(x) \bmod m_2(x) \\ \ldots \\ f(x) = s_k(x) \bmod m_k(x) \end{cases}$$

$$\rightarrow \; s(x) = f(x) \bmod m_0(x)$$

Our Ideal scheme

Lagrange Interpolation over F_p

$$s = \sum_{i=1}^{m} f(x_i) \prod_{\substack{j=1, \\ j \neq i}}^{m} \frac{x_j}{x_j - x_i} \bmod p$$

Shamir's (t,n)-SS

x_i : Public info. of shareholder U_i

$$\text{since } f(x_i) = f(x) \bmod (x - x_i), \; m_i(x) = x - x_i \in F_p$$

(Remainder Theorem for Polynomial)

Weighted (t,n)-SS based on our General Scheme

- ✓ What is Weighted (t,n)-SS
  - Each shareholder U_i in subset A has a weight w_i ;
  - secret can be recovered if

$$\sum_{i \in A} w_i \geq t$$

Weighted (t,n)-SS based on our General Scheme

✓ More natural and easier to realize Weighted (t,n)-SS based on our scheme

$$\text{weight}=\deg(m_i(x))= w_i$$

Shareholder with weight w_i is allocated a modulus polynomial of degree w_i

Conclusions

- ✓ **General (t,n)-SS Scheme** (Poly. Ring)← Asmuth-Bloom's (t,n)-SS (Integer Ring)
- ✓ **Ideal (t,n)-SS Scheme** ← General (t,n)-SS Scheme
- ✓ Shamir's scheme as a **special case** of Ideal (t,n)-SS Scheme
- ✓ Weighted **(t,n)-SS ←** General (t,n)-SS Scheme

Conclusions

Our scheme

Potential as an alternative of both schemes


Thank You
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Toda
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Kiitos
Omol
Madlobt
Ahsante
Bayarlalaa
Shukran
Tenki
Aitäh
Nandri
Köszönöm
Sobodi
Dekuji
Buznyg
Salamat
Bedankt
Supas
Dankewol
Tanemirt
Mahalo
Zikomo
Obrigada
Mèsi
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Barkal
Paldies
Aabar
Merkzi
Trugéré
Multumesc
Matondi
Blagodaram
Rahmat
Spas


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